Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Thursday 13 June 2019

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/3A**

Further Mathematics

Advanced

Paper 3A: Further Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. Use Simpson's rule with 4 intervals to estimate

$$\int_{0.4}^{2} e^{x^2}\,dx$$

(5)

$\int_a^b f(x)\,dx = \frac{1}{3}h\left[(\Sigma \text{end points}) + 4(\Sigma \text{odd values}) + 2(\Sigma \text{even values})\right]$

$h = 0.4$ with 4 intervals $\rightarrow$ 5 values (includes end point) 1

x_i	0.4	0.8	1.2	1.6	2.0
y_i	$e^{0.16} = 1.17.$	$e^{0.64} = 1.89.$	$e^{1.44} = 4.22.$	$e^{2.56} = 12.93.$	$e^{4} = 54.59..$ 1

$\int_a^b e^{x^2}\,dx = \frac{0.4}{3} \times \left[y_0 + 4(y_1 + y_3) + 2(y_2) + y_4\right]$ 1

$= \frac{0.4}{3} \times \left[1.173 + 4(1.896 + 12.935) + 2(4.22) + 54.598\right]$

$= \frac{0.4}{3} \times \left[123.54\right]$ 1

$= 16.5$ 1

P 6 1 1 7 9 A 0 2 3 2

Question 1 continued

(Total for Question 1 is 5 marks)

2. Given that k is a real non-zero constant and that

$$y = x^3 \sin kx$$

use Leibnitz's theorem to show that

$$\frac{d^5y}{dx^5} = (k^2x^2 + A)k^3x\cos kx + B(k^2x^2 + C)k^2 \sin kx$$

where A, B and C are integers to be determined.

(4)

Given $y = uv$: $\frac{d^ny}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^ku}{dx^k} \times \frac{d^{n-k}v}{dx^{n-k}}$

$y = x^3\sin(kx)$, let $u = x^3$, $v = \sin(kx)$:

$u = x^3$ $\qquad v = \sin(kx)$

$\frac{du}{dx} = 3x^2$ $\qquad \frac{dv}{dx} = k\cos(kx)$

$\frac{d^2u}{dx^2} = 6x$ $\qquad \frac{d^2u}{dx^2} = -k^2\sin(kx)$

$\frac{d^3u}{dx^3} = 6$ 1 $\qquad \frac{d^3u}{dx^3} = -k^3\cos(kx)$

can't be differentiated further $\qquad \frac{d^4u}{dx^4} = k^4\sin(kx)$

$\frac{d^5u}{dx^5} = k^5\cos(kx)$ 1

$$\frac{d^5y}{dx^5} = \sum_{k=0}^{5} \binom{5}{k} \frac{d^ku}{dx^k} \times \frac{d^{5-k}v}{dx^{5-k}}$$

$= 1(x^3 \times k^5\cos(kx)) + 5(3x^2 \times k^4\sin(kx)) + 10(6x \times -k^3\cos(kx))$

$+ 10(6 \times -k^2\sin(kx))$ 1 ← only 4 terms because $\frac{d^4u}{dx^4} = 0$

$= x^3k^5\cos kx + 15x^2k^4\sin kx - 60xk^3\cos kx$

$= (k^2x^2 - 60)k^3x\cos kx + 15(k^2x^2 - 4)k^2\sin kx$ 1

$\uparrow$ A $\qquad \uparrow$ B $\qquad \uparrow$ C

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Question 2 continued

(Total for Question 2 is 4 marks)

3.

$$\frac{dy}{dx} = x - y^2 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ay\frac{d^4y}{dx^4} + b\frac{dy}{dx}\frac{d^3y}{dx^3} + c\left(\frac{d^2y}{dx^2}\right)^2$$

where a, b and c are integers to be determined.

(4)

(b) Hence find a series solution, in ascending powers of x as far as the term in x^5, of the differential equation (I), given that $y = 1$ at $x = 0$

(5)

(a) $\frac{dy}{dx} = x - y^2$

Product rule on $-2y\frac{dy}{dx}$:

$\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ 1

$u = -2y \Rightarrow u' = -2\frac{dy}{dx}$

$\frac{d^3y}{dx^3} = -2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2$ 1 $v = \frac{dy}{dx} \Rightarrow v' = \frac{d^2y}{dx^2}$

$(u \times v') \quad (v \times u')$

Product rule on $-2y\frac{d^2y}{dx^2}$:

$u = -2y \Rightarrow u' = -2\frac{dy}{dx}$

$v = \frac{d^2y}{dx^2} \Rightarrow v' = \frac{d^3y}{dx^3}$

$uv' - vu' = -2y\frac{d^3y}{dx^3} - 2\frac{dy}{dx}\frac{d^2y}{dx^2}$

Product rule on $-2\left(\frac{dy}{dx}\right)^2$:

$u = -2\frac{dy}{dx} \Rightarrow u = -2\frac{d^2y}{dx^2}$

$v = \frac{dy}{dx} \Rightarrow v' = \frac{d^2y}{dx^2}$

$uv' - vu' = -2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\frac{d^2y}{dx^2}$

$$\frac{d^4y}{dx^4} = -2y\frac{d^3y}{dx^3} - 2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\frac{d^2y}{dx^2}$$

$$\frac{d^4y}{dx^4} = -6\frac{dy}{dx}\frac{d^2y}{dx^2} - 2y\frac{d^3y}{dx^3} \quad 1$$

Question 3 continued

Product rule on $-6\frac{dy}{dx}\frac{d^2y}{dx^2}$:

$u = -6\frac{dy}{dx} \Rightarrow u' = -6\frac{d^2y}{dx^2}$

$v = \frac{d^2y}{dx^2} \Rightarrow v' = \frac{d^3y}{dx^3}$

$uv' - vu' = -6\frac{dy}{dx}\frac{d^3y}{dx^3} - 6\left(\frac{d^2y}{dx^2}\right)^2$

Product rule on $-2y\frac{d^3y}{dx^3}$:

$u = -2y \Rightarrow u = -2\frac{dy}{dx}$

$v = \frac{d^3y}{dx^3} \Rightarrow v' = \frac{d^4y}{dx^4}$

$uv' - vu' = -2y\frac{d^4y}{dx^4} - 2\frac{dy}{dx}\frac{d^3y}{dx^3}$

$$\frac{d^5y}{dx^5} = -6\frac{dy}{dx}\frac{d^3y}{dx^3} - 6\left(\frac{d^2y}{dx^2}\right)^2 - 2y\frac{d^4y}{dx^4} - 2\frac{dy}{dx}\frac{d^3y}{dx^3}$$

$$\frac{d^5y}{dx^5} = -2y\frac{d^4y}{dx^4} - 8\frac{dy}{dx}\frac{d^3y}{dx^3} - 6\left(\frac{d^2y}{dx^2}\right)^2 \quad 1$$

(b) $y = y(0) + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \frac{x^4}{4!}\left(\frac{d^4y}{dx^4}\right)_0$

$+ \frac{x^5}{5!}\left(\frac{d^5y}{dx^5}\right)_0 \ldots$ ← Taylor series to x^5

Given $x = 0,\ y = 1$:

$\left(\frac{dy}{dx}\right)_0 = 0 - (1)^2 = -1 \quad 1$

$\left(\frac{d^2y}{dx^2}\right)_0 = 1 - 2(1)(-1) = 3$ ← using results from (a)

$\left(\frac{d^3y}{dx^3}\right)_0 = -2(1)(3) - 2(-1)^2 = -8$

$\left(\frac{d^4y}{dx^4}\right)_0 = -6(-1)(3) - 2(1)(-8) = 34$

Question 3 continued

$$\left(\frac{d^5y}{dx^5}\right)_0 = -2(1)(34) - 8(-1)(-8) - 6(3)^2 = -186$$ 2 for all 5

$$y = 1 + x(-1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(-8) + \frac{x^4}{24}(34) + \frac{x^5}{120}(-186)$$ 1

$$y = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5$$ 1

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Question 3 continued

(Total for Question 3 is 9 marks)

4. The parabola C has equation

$$y^2 = 16x$$

The distinct points $P(p^2, 4p)$ and $Q(q^2, 4q)$ lie on C, where $p \neq 0$, $q \neq 0$

The tangent to C at P and the tangent to C at Q meet at the point $R(-28, 6)$.

Show that the area of triangle PQR is 1331

(8)

$$y^2 = 16x$$

$$u = y \Rightarrow u' = \frac{dy}{dx}$$
$$v = y \Rightarrow v' = \frac{dy}{dx}$$

$$2y\frac{dy}{dx} = 16$$

$$uv' + vu' = y\frac{dy}{dx} + y\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{16}{2y} = \frac{8}{y} \quad 1$$

At P $\quad x = p^2,\ y = 4p \Rightarrow \frac{dy}{dx} = \frac{8}{4p} = \frac{2}{p} \quad 1$

$$y - 4p = \frac{2}{p}(x - p^2) \quad 1 \quad \leftarrow \quad y - y_1 = m(x - x_1)$$

$$6 - 4p = \frac{2}{p}(-28 - p^2) \quad \leftarrow \text{ R lies on tangent so use } x = -28,\ y = 6$$

$$6p - 4p^2 = -56 - 2p^2 \quad 1$$

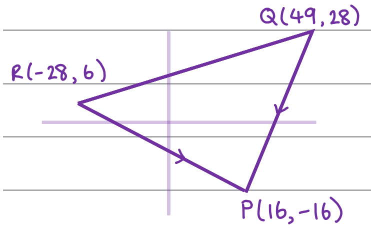
$$2p^2 - 6p - 56 = 0$$

$$(p - 7)(2p + 8) = 0$$

$$p = 7,\ -4 \quad 1$$

P and Q have the same definition and meet at the same point.

$\therefore$ point $P = (16, -16)$ and $Q = (49, 28)$ 1 (or vice versa)



$$\overrightarrow{RP} = \begin{pmatrix} 16 - -28 \\ -16 - 6 \end{pmatrix} = \begin{pmatrix} 44 \\ -22 \end{pmatrix}$$

$$\overrightarrow{QP} = \begin{pmatrix} 16 - 49 \\ -16 - 28 \end{pmatrix} = \begin{pmatrix} -33 \\ -44 \end{pmatrix}$$

$$\text{Area} = \frac{1}{2}\left|\begin{pmatrix} 44 \\ -22 \end{pmatrix} \times \begin{pmatrix} -33 \\ -44 \end{pmatrix}\right| \ 1 = \frac{1}{2}(2262) = 1331 \quad 1$$

Question 4 continued

Question 4 continued

Question 4 continued

(Total for Question 4 is 8 marks)

5.

$$I = \int \frac{1}{4\cos x - 3\sin x}\,dx \qquad 0 < x < \frac{\pi}{4}$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \frac{1}{5}\ln\left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right) + k$$

where k is an arbitrary constant.

(8)

$4\cos x - 3\sin x = 4 \times \frac{1-t^2}{1+t^2} - 3 \times \frac{2t}{1+t^2}$ 1 $\leftarrow$ $\cos x = \frac{1-t^2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

Given $t = \tan\frac{x}{2}$: $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\sec^2 = 1 + \tan^2$

$\tan x = \frac{2t}{1-t^2}$

We need to find dx in terms of t so that all the parts of the integral are in terms of t, not x.

$= \frac{1}{2}(1+t^2)$

$= \frac{1+t^2}{2}$

$dt = \frac{1+t^2}{2}\,dx$

$\frac{2dt}{1+t^2} = dx$ 1

$$\int \frac{1}{4\cos x - 3\sin x}\,dx = \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$ 1

$$= \int \frac{2}{4(1-t^2) - 3(2t)}\,dt$$

$$= \int \frac{2}{-4t^2 - 6t + 4}\,dt$$

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Question 5 continued

$$= \int \frac{-1}{2t^2 + 3t - 2}\, dt \quad (1)$$

$$= \int \frac{-1}{(2t-1)(t+2)}\, dt$$

$$\frac{-1}{(2t-1)(t+2)} = \frac{A}{(2t-1)} + \frac{B}{(t+2)}$$ ← split into partial fractions

$$-1 = A(t+2) + B(2t-1)$$

Substitute $t + 2 = 0 \Rightarrow t = -2$. ← to remove A

$$-1 = B(2 \times -2 - 1) \Rightarrow B = \frac{1}{5}$$

Substitute $2t - 1 = 0 \Rightarrow t = \frac{1}{2}$. ← to remove B

$$-1 = A\left(\frac{1}{2} + 2\right) \Rightarrow A = -\frac{2}{5}$$ (1) for A, B

be careful of this!
$-5(2t-1) = 5(1-2t)$

$$\int \frac{-1}{(2t-1)(t+2)}\, dt = \int \frac{2}{5(1-2t)} + \frac{1}{5(t+2)}\, dt$$

$$= \frac{1}{5}\int \frac{2}{(1-2t)} + \frac{1}{(t+2)}\, dt \quad (1)$$

$\int \frac{f'(x)}{f(x)}\, dx = \ln(f(x)) + c$

$$= \frac{1}{5}\ln(t+2) - \frac{1}{5}\ln(1-2t) + k \quad (1)$$

$$= \frac{1}{5}\ln\left(\frac{t+2}{1-2t}\right) + k$$

$$= \frac{1}{5}\ln\left(\frac{2 + \tan\frac{x}{2}}{1 - 2\tan\frac{x}{2}}\right) + k \quad (1)$$

Question 5 continued

Question 5 continued

(Total for Question 5 is 8 marks)

6. The concentration of a drug in the bloodstream of a patient, t hours after the drug has been administered, where $t \leqslant 6$, is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2C}{\mathrm{d}t^2} - 5t\frac{\mathrm{d}C}{\mathrm{d}t} + 8C = t^3 \qquad \text{(I)}$$

where C is measured in micrograms per litre.

(a) Show that the transformation $t = \mathrm{e}^x$ transforms equation (I) into the equation

$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} - 6\frac{\mathrm{d}C}{\mathrm{d}x} + 8C = \mathrm{e}^{3x} \qquad \text{(II)}$$

(5)

(b) Hence find the general solution for the concentration C at time t hours.

(7)

Given that when $t = 6$, $C = 0$ and $\frac{\mathrm{d}C}{\mathrm{d}t} = -36$

(c) find the maximum concentration of the drug in the bloodstream of the patient.

(5)

(a) $t = e^x \implies \frac{dt}{dx} = t$ and $\frac{d^2t}{dx^2} = t$

Using the chain rule : $\frac{dC}{dt} = \frac{dC}{dx} \times \frac{dx}{dt}$

$= \frac{dC}{dx} \times \frac{1}{t}$ (1)

$\frac{dC}{dx} = t \times \frac{dC}{dt} \implies \frac{d}{dx}\left(\frac{dC}{dx}\right) = \frac{d}{dx}\left(t \times \frac{dC}{dt}\right)$

Product rule on $t \times \frac{dC}{dt}$ w.r.t x :

$u = t \qquad v = \frac{dC}{dt}$

$u' = \frac{dt}{dx} \qquad v' = \frac{d}{dx}\left(\frac{dC}{dt}\right)$

$uv' + vu' = t \times \frac{d}{dx}\left(\frac{dC}{dt}\right) + \frac{dt}{dx} \times \frac{dC}{dt}$

$\frac{d^2C}{dx^2} = t \times \frac{d}{dx}\left(\frac{dC}{dt}\right) + t \times \frac{dC}{dt}$

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Question 6 continued

Using the chain rule again: $\frac{d}{dx}\left(\frac{dC}{dt}\right) = \frac{d}{dt}\left(\frac{dC}{dt}\right) \times \frac{dt}{dx}$

$= \frac{d^2C}{dt^2} \times t$

$\therefore \frac{d^2C}{dx^2} = t \times \frac{dC}{dt} + t\left(\frac{d^2C}{dt^2} \times t\right)$

$= t\frac{dC}{dt} + t^2\frac{d^2C}{dt^2}$ $\times \frac{1}{t}$

$\frac{d^2C}{dx^2} \times \frac{1}{t} = \frac{dC}{dt} + t\frac{d^2C}{dt^2}$ 2

$\frac{d^2C}{dx^2} \times \frac{1}{t} = \frac{1}{t} \times \frac{dC}{dx} + t\frac{d^2C}{dt^2}$

$\frac{d^2C}{dx^2} - \frac{dC}{dx} = t^2 \times \frac{d^2C}{dt^2}$

Taking the original equation:

$t^2\frac{d^2C}{dt} - 5t\frac{dC}{dt} + 8C = \frac{d^2C}{dx^2} - \frac{dC}{dx} - 5t \times \frac{1}{t}\left(\frac{dC}{dt}\right) + 8C$ 1

$e^{3x} = \frac{d^2C}{dx^2} - 6\frac{dC}{dx} + 8C$ 1

Question 6 continued

(b) Find the general solution:

$$\frac{d^2C}{dx^2} - 6\frac{dC}{dx} + 8c = 0 \quad \leftarrow \text{set equal to } 0$$

$$m^2 - 6m + 8 = 0 \quad \leftarrow \text{characteristic equation}$$

$$(m-4)(m-2) = 0 \Rightarrow m = 2, 4 \quad 1$$

There are two positive integer roots, so homogenous solution has the form:

$$C_h = Ae^{4x} + Be^{2x} \quad 1$$

Find the particular integral:

Assume a solution $C_p = ke^{3x}$ 1 $\Rightarrow \frac{dC}{dx} = 3ke^{3x}$

$$\Rightarrow \frac{d^2C}{dx^2} = 9ke^{3x}$$

$$\underline{9ke^{3x}} - 6(\underline{3ke^{3x}}) + 8(\underline{ke^{3x}}) = e^{3x}$$

substitute into the differential equation

$$9k - 18k + 8k = 1$$

$$k = -1 \quad 1$$

C = characteristic equation + particular integral

$$C = Ae^{4x} + Be^{2x} - 1(e^{3x}) \quad 1$$

$$t = e^{x} \quad 1 \quad \Rightarrow C = At^4 + Bt^2 - t^3 \quad 1$$

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P 6 1 1 7 9 A 0 2 0 3 2

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Question 6 continued

(c) When $t = 6$, $C = 0$:

$$0 = A(6^4) + B(6^2) - (6^3)$$

$$0 = 1296A + 36B - 216 \quad ①$$

$$\frac{dC}{dt} = 4At^3 + 2Bt - 3t^2$$

When $t = 6$, $\frac{dC}{dt} = -36$:

$$-36 = 4A(6^3) + 2B(6) - 3(6^2)$$

$$-36 = 864A + 12B - 108 \quad ①$$

$$\left.\begin{array}{l} 1296A + 36B = 216 \\ 864A + 12B = 72 \end{array}\right\}$$ solve simultaneously to get: $A = 0$, $B = 6$

$\therefore C = 6t^2 - t^3$ ① and $\frac{dC}{dt} = 12t - 3t^2$

$12t - 3t^2 = 0 \leftarrow$ max. when $\frac{dC}{dt} = 0$

$t(12 - 3t) = 0$

$12 - 3t = 0 \Rightarrow t = 4$ ①

$C = 6(4)^2 - (4)^3 = 32\ \mu g L^{-1}$ ①

Question 6 continued

Question 6 continued

(Total for Question 6 is 17 marks)

7. With respect to a fixed origin O, the points A, B and C have coordinates (3, 4, 5), (10, –1, 5) and (4, 7, –9) respectively.

The plane Π has equation $4x - 8y + z = 2$

The line segment AB meets the plane Π at the point P and the line segment BC meets the plane Π at the point Q.

(a) Show that, to 3 significant figures, the area of quadrilateral $APQC$ is 38.5

(6)

The point D has coordinates $(k, 4, -1)$, where k is a constant.

Given that the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ form three edges of a parallelepiped of volume 226

(b) find the possible values of the constant k.

(4)

(a) Area APQC = Area ABC − Area PBQ 1

Area ABC = $\frac{1}{2}|AB \times BC|$, Area PBQ = $\frac{1}{2}|PB \times BQ|$

Equation of a line through a, parallel to b is $r = a + \lambda b$:

$$AB: \quad r = \underset{(A)}{\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}} + \lambda \underset{(B-A)}{\begin{pmatrix} 10-3 \\ -1-4 \\ 5-5 \end{pmatrix}} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix}$$

$$BC: \quad r = \underset{(B)}{\begin{pmatrix} 10 \\ -1 \\ 5 \end{pmatrix}} + \lambda \underset{(B-C)}{\begin{pmatrix} 10-4 \\ -1-7 \\ 5+9 \end{pmatrix}} = \begin{pmatrix} 10 \\ -1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ -8 \\ 14 \end{pmatrix} \quad 1$$

P and Q are where AB and BC meet the plane Π:

$4(3+7\lambda) - 8(4-5\lambda) + 5 = 2$ 1 method to find P or Q

$12 + 28\lambda - 32 + 40\lambda + 5 = 2$

$$\lambda = \frac{1}{4} \Rightarrow P = \begin{pmatrix} 3+\frac{7}{4} \\ 4-\frac{5}{4} \\ 5 \end{pmatrix} = \begin{pmatrix} 4.75 \\ 2.75 \\ 5 \end{pmatrix}$$

Question 7 continued

$$4(10+6\mu)-8(-1-8\mu)+5+14\mu=2$$

$$40+24\mu+8+64\mu+5+14\mu=2$$

$$\mu=-\frac{1}{2}$$

$$Q=\begin{pmatrix}10-\frac{6}{2}\\-1-\frac{8}{2}\\5-\frac{14}{2}\end{pmatrix}=\begin{pmatrix}7\\3\\-2\end{pmatrix}$$ ① P AND Q correct

$$\text{Area } ABC=\frac{1}{2}\begin{vmatrix}i & j & k\\7 & -5 & 0\\6 & -8 & 14\end{vmatrix}\begin{array}{l}\\ \leftarrow \text{direction of } AB\\ \leftarrow \text{direction of } BC\end{array}$$

$$=\frac{1}{2}\left[(-5\times14-0\times-8)i-(7\times14-0\times6)j+(7\times-8--5\times6)k\right]$$

$$=\frac{1}{2}\left[70i-98j+26k\right]$$

$$=\frac{1}{2}\sqrt{70^2+98^2+26^2}$$ ①

$$=61.604$$

$$\overrightarrow{PB}=(10-4.75)i+(-1-2.75)j+(5-5)k=5.25i-3.75j$$

$$\overrightarrow{QB}=(10-7)i+(-1-3)j+(5--2)k=3i-4j+7k$$

$$\text{Area } PBQ=\frac{1}{2}\begin{vmatrix}i & j & k\\5.25 & -3.75 & 0\\3 & -4 & 7\end{vmatrix}\begin{array}{l}\\ \leftarrow \text{direction of } PB\\ \leftarrow \text{direction of } QB\end{array}$$

$$=\frac{1}{2}\left[(-3.75\times7--4\times0)i-(5.25\times7-3\times0)j+(5.25\times-4-3.75\times3)k\right]$$

$$=\frac{1}{2}\left[26.25i-36.75j-9.75k\right]$$

Question 7 continued

$= \frac{1}{2}\sqrt{26\ 25^2 + 36.75^2 + 9.75^2}$

$= 23\ 101$

Area $APQC = 61.604 - 23.101 = 38.5$ ①

(b) $\overrightarrow{AB} = B - A = \begin{bmatrix} 7 \\ -5 \\ 0 \end{bmatrix}$

$\overrightarrow{AC} = C - A = \begin{bmatrix} 4-3 \\ 7-4 \\ -9-5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -14 \end{bmatrix}$

$\overrightarrow{AD} = D - A = \begin{bmatrix} k-3 \\ 4-4 \\ -1-5 \end{bmatrix} = \begin{bmatrix} k-3 \\ 0 \\ -6 \end{bmatrix}$ ①

$\overrightarrow{AB} \times (\overrightarrow{AC} \cdot \overrightarrow{AD}) = \begin{vmatrix} 7 & -5 & 0 \\ 1 & 3 & -14 \\ k-3 & 0 & -6 \end{vmatrix}$

$\pm 226 = 7(3 \times -6) - 5(1 \times -6 - -14(k-3))$

$\pm 226 = -126 + 5(-6 + 14k - 42)$

$\pm 226 = -126 - 30 + 70k - 210$

$\pm 226 = -366 + 70k$

$-226 + 366 = 70k$ $\qquad$ $226 + 366 = 70k$

$k = 2$ $\qquad$ $k = \frac{296}{35}$

Question 7 continued

(Total for Question 7 is 10 marks)

8. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

The line l_1 is the tangent to H at the point $P(4\cosh\theta, 3\sinh\theta)$.

The line l_1 meets the x-axis at the point A.

The line l_2 is the tangent to H at the point $(4, 0)$.

The lines l_1 and l_2 meet at the point B and the midpoint of AB is the point M.

(a) Show that, as θ varies, a Cartesian equation for the locus of M is

$$y^2 = \frac{9(4 - x)}{4x} \qquad p < x < q$$

where p and q are values to be determined.

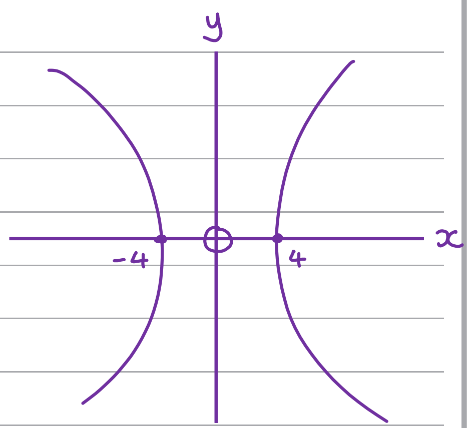
(11)

Let S be the focus of H that lies on the positive x-axis.

(b) Show that the distance from M to S is greater than 1

(3)

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ differentiate w.r.t x.



$\frac{x}{8} - \frac{2y \times \frac{dy}{dx}}{9} = 0$

$\frac{x}{8} = \frac{2y\frac{dy}{dx}}{9}$ $\times 9$

$\frac{9x}{8} = 2y\frac{dy}{dx}$ $\div 2y$

$\frac{9x}{16y} = \frac{dy}{dx}$ 1

When $x = 4\cosh\theta$, $y = 3\sinh\theta$. ← l_1 is the tangent to H at point P.

$\frac{dy}{dx} = \frac{9(4\cosh\theta)}{16(3\sinh\theta)}$

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Question 8 continued

$$\frac{dy}{dx} = \frac{3\cosh\theta}{4\sinh\theta}$$

Equation of l_1 : $y - y_1 = m(x - x_1)$

$$y - 3\sinh\theta = \frac{3\cosh\theta}{4\sinh\theta}(x - 4\cosh\theta) \quad 1$$

Point A is on the x-axis, so $y = 0$:

$$0 = \frac{3\cosh\theta}{4\sinh\theta}(x - 4\cosh\theta) + 3\sinh\theta$$

$$-3\sinh\theta = \frac{3\cosh\theta}{4\sinh\theta}(x - 4\cosh\theta)$$

$$\div \frac{3\cosh\theta}{4\sinh\theta}$$

$$\frac{(-3\sinh\theta)(4\sinh\theta)}{3\cosh\theta} = x - 4\cosh\theta$$

$$+ 4\cosh\theta$$

$$\frac{-12\sinh^2\theta}{3\cosh\theta} + 4\cosh\theta = x$$

$$\frac{-4\sinh^2 + 4\cosh^2\theta}{\cosh\theta} = x$$

$$\cosh^2\theta - \sinh^2\theta \equiv 1$$

this is the point A → $\frac{4}{\cosh\theta} = x$ 1

Equation of l_2 : $x = 4$ 1 where l_1 meets l_2

$$y - 3\sinh\theta = \frac{3\cosh\theta}{4\sinh\theta}(4 - 4\cosh\theta) \quad 1$$

$$y = \frac{12\cosh\theta - 12\cosh^2\theta}{4\sinh\theta} + 3\sinh\theta$$

Question 8 continued

$$y = \frac{3\cosh\theta - 3\cosh^2\theta + 3\sinh^2\theta}{\sinh\theta}$$

$$y = \frac{3\cosh\theta - 3(\cosh^2\theta - \sinh^2\theta)}{\sinh\theta}$$

this is the point B. $\rightarrow$ $y = \dfrac{3\cosh\theta - 3}{\sinh\theta}$ ① $\quad \cosh^2\theta - \sinh^2\theta \equiv 1$

$$A = \left(\frac{4}{\cosh\theta}, 0\right) \text{ and } B = \left(4, \frac{3\cosh\theta - 3}{\sinh\theta}\right)$$

$$\text{Midpoint} = \left[\frac{1}{2}\left(4 + \frac{4}{\cosh\theta}\right), \frac{1}{2}\left(\frac{3\cosh\theta - 3}{\sinh\theta}\right)\right] \text{ ①}$$

$$x = \frac{1}{2}\left(4 + \frac{4}{\cosh\theta}\right) = 2 + \frac{2}{\cosh\theta}$$

$$\therefore \cosh\theta = \frac{2}{x-2}$$

$$y = \frac{1}{2}\left(\frac{3\cosh\theta - 3}{\sinh\theta}\right) = \frac{3\cosh\theta - 3}{2\sinh\theta}$$

$$y^2 = \frac{9(\cosh\theta - 1)^2}{4\sinh^2\theta} = \frac{9\left(\frac{2}{x-2} - 1\right)^2}{4\left(\frac{2}{x-2} + 1\right)^2} \text{ ①}$$

simplify top and bottom

$$y^2 = \frac{9\left(\frac{4-x}{x-2}\right)^2}{4\left(\frac{x}{x-2}\right)^2}$$

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Question 8 continued

$$y^2 = \frac{9(4-x)}{4x} \quad ①$$

$(x-2)$ cancels

$x < 4$ because $9(4-x) > 0$ to make $y^2 > 0$

$x > 2$ because $\frac{x}{x-2} > 0$ to make $y^2 > 0$

$\therefore \quad p = 2$ ①, $q = 4$ ①

(b) Focus of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(\pm ae, 0)$ where $b^2 = a^2(e^2-1)$

$b^2 = a^2(e^2-1) \quad \Rightarrow \quad 9 = 16(e^2-1)$

$$\frac{9}{16} + 1 = e^2$$

$$\sqrt{\frac{25}{16}} = e$$

$$\frac{5}{4} = e \quad ①$$

Focus is at $x = +ae = 4 \times \frac{5}{4} = 5$ ①

Distance $d > 5 - 4 \qquad \therefore \ d > 1$ ①

Question 8 continued

(Total for Question 8 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS